Lecture Mar 31: Cubic equations

Course announcements

- HW 1 due Friday
  ($\mathbb{F}_3$ notation now explained) p prime $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/(p)$
  ~~Hungerford uses~~ $\mathbb{Z}_p$

(for any field k, $\underbrace{k[x,y]}_{\text{commuting}}$ = ring of poly. in x & y

Ex: $\frac{x}{y} \in k(x,y)$

Notation: $k\langle x,y\rangle$ = non-comm. ring $xy \neq yx$
$k(x,y)$ = field of rational polynomials $\frac{f(x,y)}{g(x,y)}$ where
$f, g \in k[x,y]$ and $g \neq 0$.

- Office hours
  - Today 3–4 pm (zoom link in email)
- Reflection #1 due today
  - Canvas, not gradescope
- Quizzes on Monday (1st one is April 12) Total of 7 quizzes

§1. Review of quadratic eqns

How can we solve $f(x) = \boxed{ax^2+bx+c=0}$?

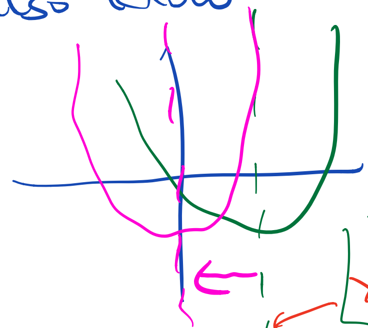
"Completing the square"

- If $a=0$, this is linear. We know $x=-c/b$ is a soln.

$\boxed{\text{So we can assume } a\neq 0}$

- If $c=0$, also know how to solve.

$\frac{df}{dx} = 2ax+b$



$\boxed{x=\frac{-b}{2a}}$

Dividing out by a, we can assume equation is

$$x^2+bx+c$$

Min/max occurs at $\frac{-b}{2}$!

- If $b=0$, also easy! $x^2+c=0 \Rightarrow x=\pm\sqrt{-c}$

Can we arrange somehow for $b=0$?

Linear substitution

$$x \rightsquigarrow x+\alpha \quad \text{for constant } \alpha$$

$$x^2+bx+c \rightsquigarrow (x+\alpha)^2+b(x+\alpha)+c =$$

$$x^2+2\alpha x+\alpha^2+bx+b\alpha+c =$$

$$x^2+\underbrace{(2\alpha+b)}x+\alpha^2+b\alpha+c$$

For this to be 0, need $\alpha=\frac{-b}{2}$

But the geometry of the parabola told us this!

$$\rightsquigarrow \underbrace{f\left(x-\frac{b}{2}\right)}_{g(x)} = x^2+\frac{b^2}{4}+\frac{-b^2}{2}+c$$

$$= x^2-\frac{b^2}{4}+c$$

Know, $y=\pm\sqrt{\frac{b^2}{4}-c}$ is a soln to g

$$\rightsquigarrow x=\frac{-b}{2}\pm\sqrt{\frac{b^2}{4}-c} \quad \text{soln for } f$$

$$x = \frac{-b}{2} \pm \sqrt{\frac{b^2}{4} - c}$$ soln for $f = x^2 + bx + c$

To solve $a_2x^2 + a_1x + a_0 = 0$

$$\to \boxed{\frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}}$$

Babylonian's 2000 BC

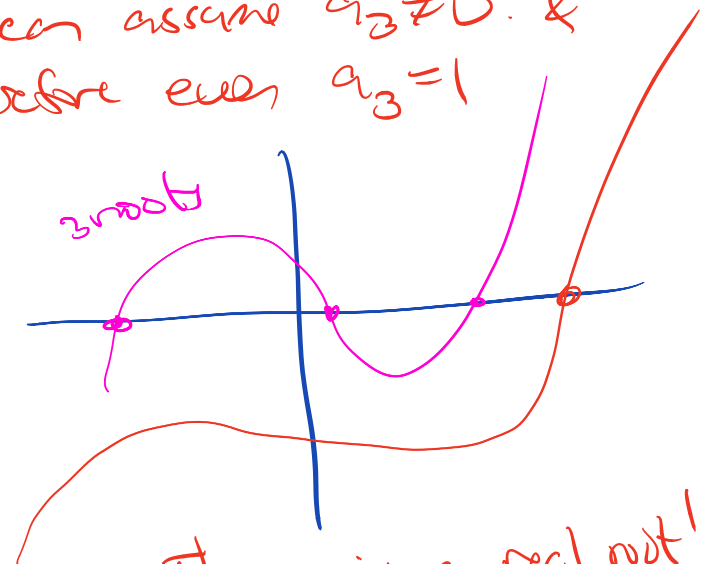
completely discarded complex solns as absurd.

§2. Cubic equation

How do we solve $a_3x^3 + a_2x^2 + a_1x + a_0 = 0$?

- If $a_3 = 0$, ✓
- We can assume $a_3 \neq 0$. & therefore even $a_3 = 1$

Cubic



There is a real root!

Linear substitution $x \to x + \alpha$

$$(x+\alpha)^3 + a_2(x+\alpha)^2 + a_1(x+\alpha) + a_0$$

$$= x^3 + (3\alpha + a_2)x^2 + (\quad)x + (\quad)$$

$$\boxed{\alpha = -a_2/3}$$

Can assume

$$\boxed{f(x) = x^3 + a_1 x + a_0 = 0}$$

Cannot after linear change get $a_1 = 0$

New tricks: Substitute

$$x = y - \frac{a_1}{3y}$$

$$\rightarrow f(x) = \left(y - \frac{a_1}{3y}\right)^3 + a_1\left(y - \frac{a_1}{3y}\right) + a_0 = 0$$

$$= \left(y^3 - \cancel{a_1 y} + \cancel{\frac{a_1^2}{3y}} - \frac{a_1^3}{27y^3}\right) + \cancel{a_1\left(y - \frac{a_1}{3y}\right)} + a_0 = 0$$

$$= y^3 - \frac{a_1^3}{27y^3} + a_0 = 0$$

Multiply by $\boxed{y^3}$

$$= y^6 + a_0 y^3 - \frac{a_1^3}{27} = 0$$

$$= (y^3)^2 + a_0(y^3) - \frac{a_1^3}{27} = 0$$

Quadratic formula

$$y^3 = \frac{-a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}$$

$$\rightarrow \boxed{y = \sqrt[3]{\frac{-a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}}}$$

$\rightarrow$ Then need to go back and $x = y - \frac{a_1}{3y}$

Keep in mind that there a 3 cube roots

$\omega = e^{2\pi i/3}$ then $\omega^3 = 1$

$$y = \sqrt[3]{\frac{-a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}}$$

$$\omega\sqrt[3]{\frac{-a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}}$$

$$\omega^2\sqrt[3]{\frac{-a_0}{2} \pm \sqrt{\frac{a_0^2}{4} + \frac{a_1^3}{27}}}$$

6 sol'n but only 3 are sol'ns